Unit 5 – Reciprocal Functions
5.1 – Graphs of Reciprocal Functions

If \( y = f(x) \), its reciprocal is \( y = \frac{1}{f(x)} \)

Properties of Reciprocal Functions

- There will be vertical asymptotes at the zeros of the original function;
- There will be a horizontal asymptote where \( y = 0 \);
- The original function and its reciprocal have the same **positive** and **negative** intervals (above and below the x-axis);
- The intervals of increase and decrease switch;
- The y-coordinates of the reciprocal functions are reciprocals (1/#) of the original function;
- The original function and the reciprocal function will intersect if \( f(1) = \pm 1 \);
- If the original function has a local minimum, the reciprocal has a local maximum at the same x-value (and vice versa).

Examples:

i) Linear
\[ y = x + 3 \]

ii) Quadratic w/2 Zeros
\[ y = x^2 - 4 \]
Why are there vertical asymptotes in the reciprocal function at the zeros of the original function?

- Reciprocal is undefined \( (x,0) \rightarrow (x, \frac{1}{0}) \)

Why is there always a horizontal asymptote at \( y = 0 \) for a reciprocal function?

- Reciprocal is \( y = \frac{1}{f(x)} \)

- Can never equal 0

How do we graph a reciprocal function?

- Draw in vertical asymptotes (if any)
- Use known points (y-intercepts, vertex, etc.) to find points on the reciprocal
- Use intervals of original function to determine intervals of reciprocal

Let’s take a look at some functions... you have 30 minutes to graph!

Try Pg. 254 #1,2,4,5,7,8abc,9,10,12
5.2 – Exploring Quotients of Polynomial Functions

Rational Function

- A function expressed as \( f(x) = \frac{p(x)}{q(x)} \)

Where \( p(x) \) and \( q(x) \) are polynomial functions and \( q(x) \neq 0 \)

Example: \( f(x) = \frac{3x - 1}{x^2}, \ x \neq 0 \)

Hole – a point where a function is undefined

Oblique Asymptote – a slanted asymptote

Rational Functions Review:

- To determine the location of discontinuities in rational functions, make sure you factor the denominator fully
- State the restrictions on your rational function
- Factor you numerator as well
- Factors that are common to your numerator and denominator result in holes in your graph (0/0)
- Factors that are unique to the denominator result in asymptotes

Example: Simplify. State all restrictions.

\[
f(x) = \frac{3x - 9}{x^2 - 9}
\]

\[
f(x) = \frac{3(x - 3)}{(x + 3)(x - 3)}
\]

\[
f(x) = \frac{3}{x + 3}, \quad x \neq \pm 3
\]

Hole @ \( x = 3 \)
V.A. @ \( x = -3 \)
Characteristics of Rational Functions

- A rational function, \( f(x) = \frac{p(x)}{q(x)} \), will have a hole at \( x = a \) if \( p(a) = 0 \) and \( q(a) \neq 0 \). This happens if there is a common factor, \( (x - a) \), in both the numerator and denominator.

- A rational function will have a vertical asymptote at \( x = a \) if \( p(a) \neq 0 \) and \( q(a) = 0 \).

- A rational function, \( f(x) = \frac{p(x)}{q(x)} \), will have a horizontal asymptote if the degree of \( p(x) \) is less than or equal to the degree of \( q(x) \). To understand IF it has a horizontal asymptote, we will only have to consider the ratio of the leading coefficients. For example, \( f(x) = \frac{2x}{x + 1} \).

- A rational function, \( f(x) = \frac{p(x)}{q(x)} \), will have an oblique asymptote if the degree of \( p(x) \) is greater than the degree of \( q(x) \) by exactly 1. We will only have to understand IF it has an oblique asymptote. For example, \( f(x) = \frac{x^2 + 4}{x + 1} \).

Example: Simplify the following rational functions. State the locations of any holes or vertical asymptotes. State the existence of any oblique or horizontal asymptotes.

a) \( f(x) = \frac{x^2 - 4}{3x + 6} \)

b) \( g(x) = \frac{2x + 3}{x - 2} \)

Try Pg. 262 #1,2
5.3 – Graphs of Rational Functions in the Form:

\[ f(x) = \frac{ax + b}{cx + d} \]

Most rational functions in this form will have both a H.A. and V.A. To sketch these graphs:

- Use the vertical asymptotes and x-intercepts to create a factor table (will allow you to identify the positive and negative intervals)
- The end behaviours will approach the HORIZONTAL ASYMPTOTE, unless there is not one

Vertical Asymptote: \[ x = \frac{-d}{c} \]  
Horizontal Asymptote: \[ y = \frac{a}{c} \]

Ex. Graph \[ f(x) = \frac{4x + 1}{2x - 3} \]

V.A. @ \( x = \) ___

H.A. @ \( y = \) ___

x-int =

Use a factor table using the H.A. and x-int to create your intervals:

Use test points to finish graphing your function:

| \( x \) | \( y \) |
| -1 |     |
| 0 |     |
| 1 |     |

| \( x \) | \( y \) |
| 2 |     |
| 3 |     |
| 4 |     |
When the numerator and denominator have a common factor, you end up with a graph of a horizontal line with a hole at the zero of the common factor.

**Ex.** Graph \( f(x) = \frac{6x - 12}{4x - 8} \)

\[ f(x) = \frac{6(x - 2)}{4(x - 2)} \]

\[ f(x) = \frac{3}{2}, \ x \neq 2 \]

**Example:** For \( f(x) = \frac{x + 4}{4x - 8} \)

State the domain, intercepts, asymptotes, and positive and negative intervals. Sketch the graph of the function and state where it is increasing or decreasing, as well as the end behaviours of the function.

Try Pg. 272 #1-3,5,6,8-12,14abc